TOTAL SINGULARITY DEGREE 4, CASE 1+3

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Let q and q' be members of Q. Let c and c' be members of a cyclic cubic field K. The Galois group of K is C_3, which we will write multiplicatively. Let C_3 be generated by g and denote the action of g (resp. g^2) on c by c^* (resp. c^{**}). By analogy with the cross-ratio of four rational values, define an invariant I_1 which is a function of q and c as follows:

$$I_1(q, c) = \frac{(q - c)(c^* - c^{**})}{(q - c^{**})(c^* - c)}$$

Using the embedding of the cyclic cubic field into the algebra, express the elements c and c' as follows:

$$c = z + rJ(\theta)$$

$$c' = z' + r'J(\theta')$$

The embedding was chosen so that the Galois conjugates of c and c' can be expressed easily. For example:

$$c^* = z + rJ(\theta + \frac{2\pi}{3})$$

$$c^{**} = z + rJ(\theta + \frac{4\pi}{3})$$

The Galois conjugates of c' are similar. The invariant I_1 then assumes this form:

$$I_1(q, c) = \frac{(q - z - rJ(\theta))(J(\theta + \frac{2\pi}{3}) - J(\theta + \frac{4\pi}{3}))}{(q - z - rJ(\theta + \frac{4\pi}{3}))(J(\theta + \frac{2\pi}{3}) - J(\theta))}$$

Equate the invariant I_1 evaluated at (q, c) and (q', c'):

$$I_1(q, c) = I_1(q', c')$$

$I_1(q', c')$ has this value:

$$I_1' = \frac{(q' - z' - r'J(\theta'))(J(\theta' + \frac{2\pi}{3}) - J(\theta' + \frac{4\pi}{3}))}{(q' - z' - r'J(\theta' + \frac{4\pi}{3}))(J(\theta' + \frac{2\pi}{3}) - J(\theta'))}$$

Multiply through by the denominators in the two expressions for $I_1(q, c)$ and $I_1(q', c')$ to get this equation:

$$LHS = (q - z - rJ(\theta))(J(\theta + \frac{2\pi}{3}) - J(\theta + \frac{4\pi}{3}))(q' - z' - r'J(\theta' + \frac{4\pi}{3}))(J(\theta' + \frac{2\pi}{3}) - J(\theta'))$$