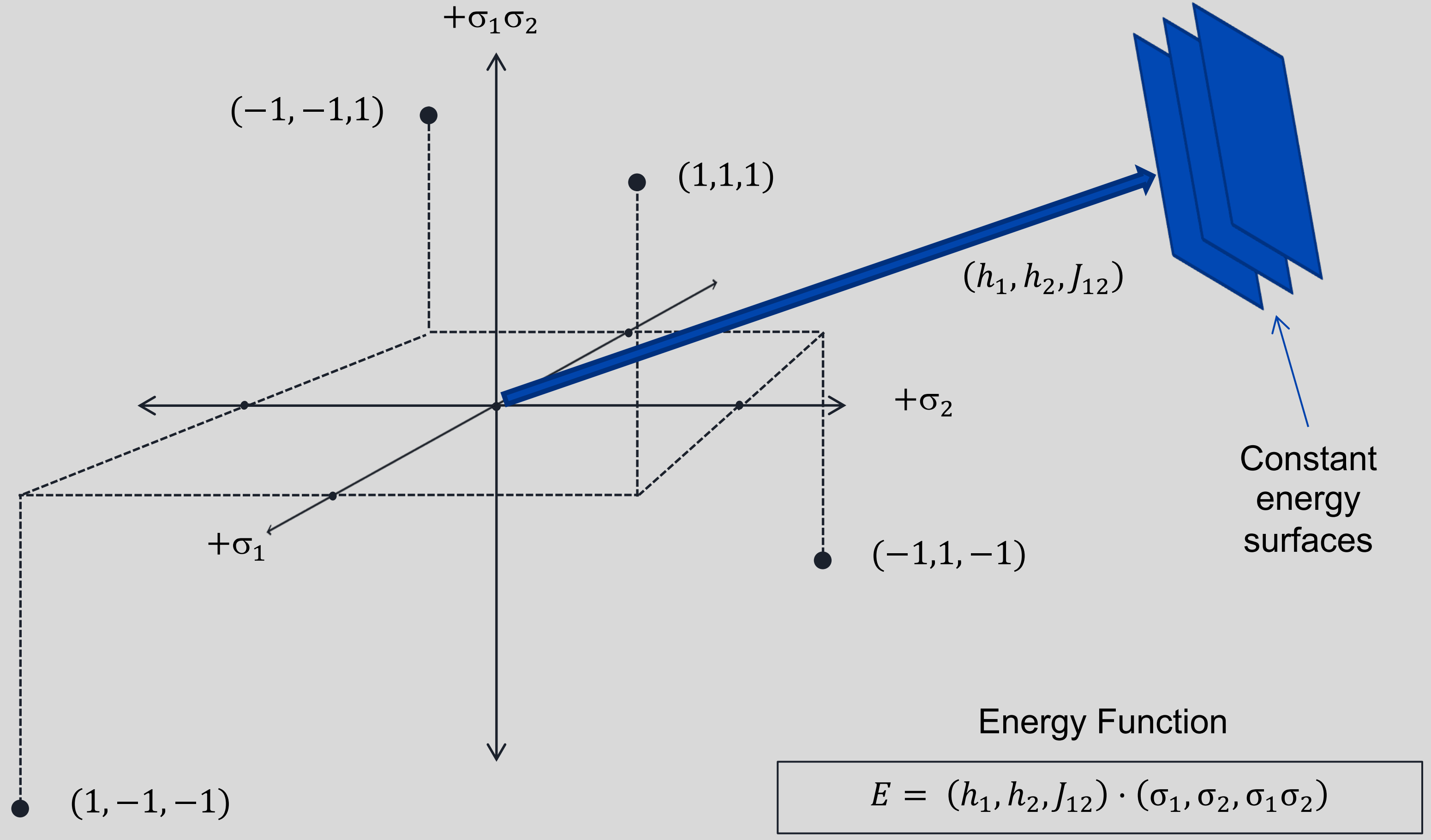
QUBOs
Polytopes
Symmetry and all that

Denny Dahl

This talk will ...

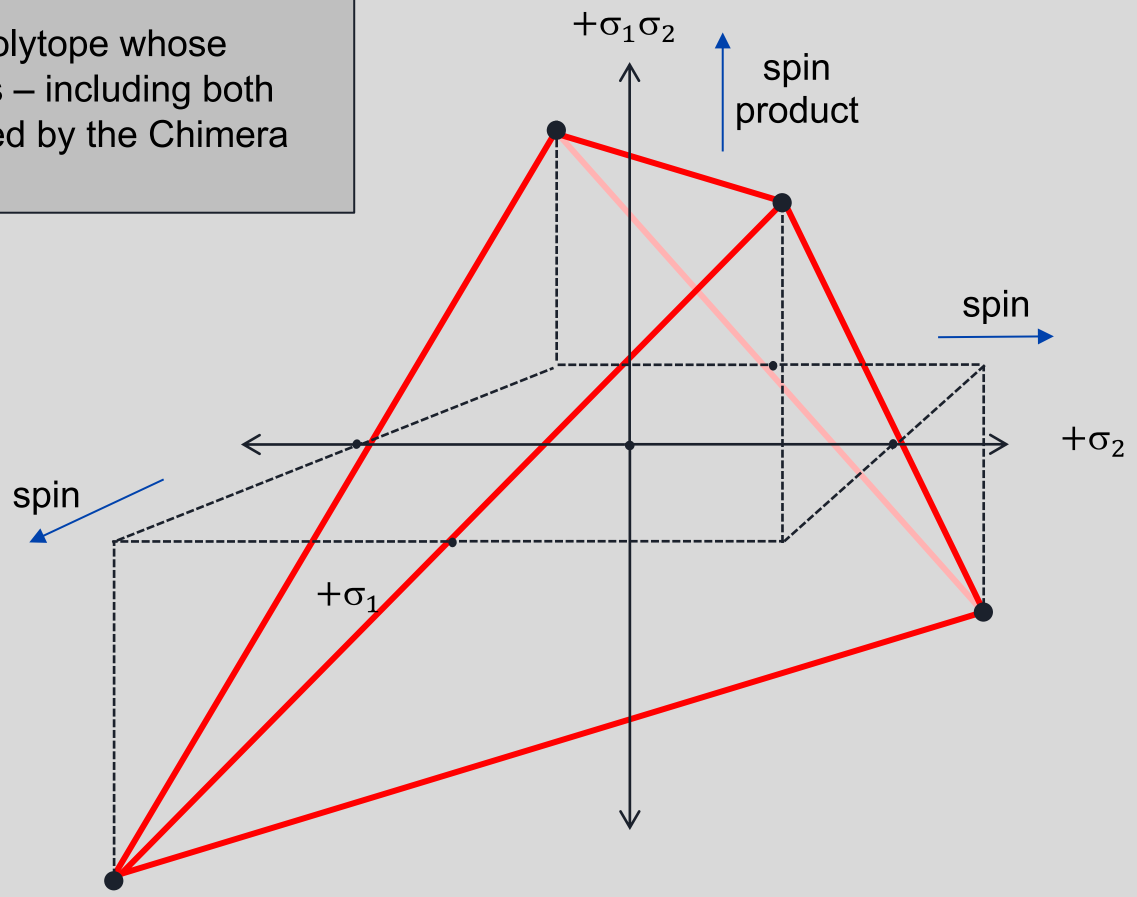
- ... introduce a simple example
- ... define the Chimeratope
- ... establish a connection with geometry
- ... assert that solving Ising models is equivalent to Linear Programming
- ... show how symmetries in an Ising model are reflected in geometric symmetries
- ... describe the symmetries of some small Chimeratopes
- ... argue for deformation of Ising models
- ... introduce the midpoint algorithm



$+\sigma_1\sigma_2$
$(-1,-1,1)$
$(1,1,1)$
(h_1, h_2, J_{12})
$+\sigma_2$
Constant energy surfaces
$+\sigma_1$
$(-1,1,-1)$
$(1,-1,-1)$
Energy Function
$E = (h_1, h_2, J_{12}) \cdot (\sigma_1, \sigma_2, \sigma_1\sigma_2)$

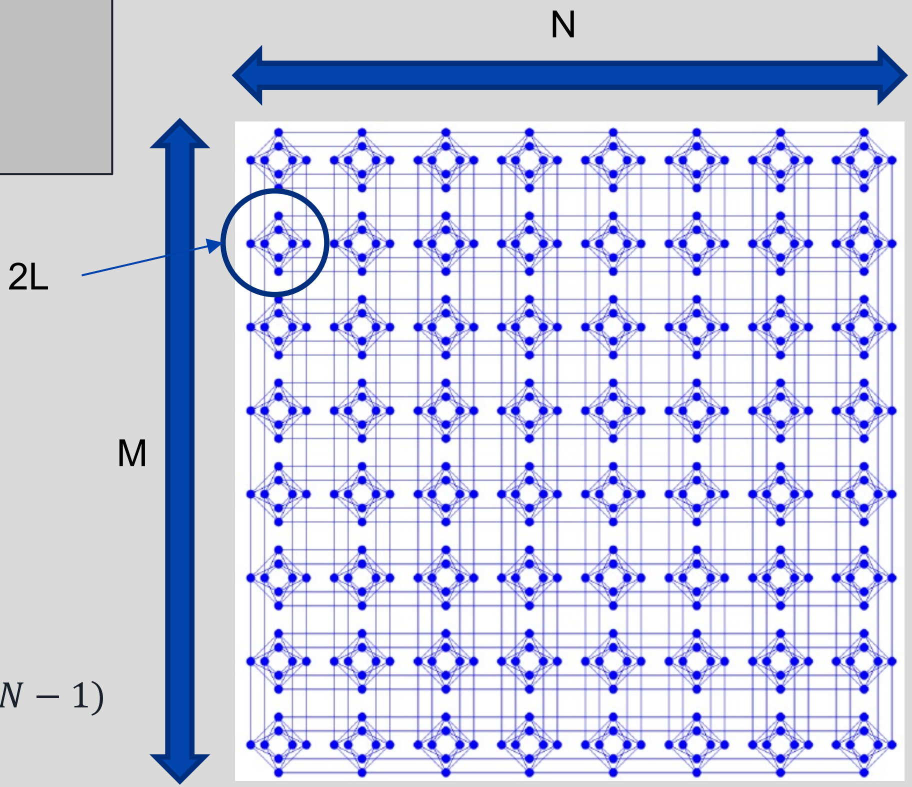
Chimeratope

The **Chimeratope** is the convex polytope whose vertices are the spin configurations – including both linear and quadratic terms – allowed by the Chimera topology.



Chimeratope(L,M,N)

- L = half number of spins in the unit cell
- M = number of rows
- N = number of columns



Total number of qubits: $2LMN$

Number of h_i coefficients: $2LMN$

Number of intracell J_{ij} coefficients: L^2MN

Number of intercell J_{ij} coefficients: $L(M-1)N + LM(N-1)$

Chimeratope has 2^{2LMN} vertices

Chimeratope is $L^2MN + L(M-1)N + LM(N-1)$-dimensional

**************** Wednesday December 19, 2012 ****************

In Sudbury

Here is an article from the Simons Foundation:

https://simonsfoundation.org/tag/quantum-computing/

This blog mentions an article which seems very interesting:

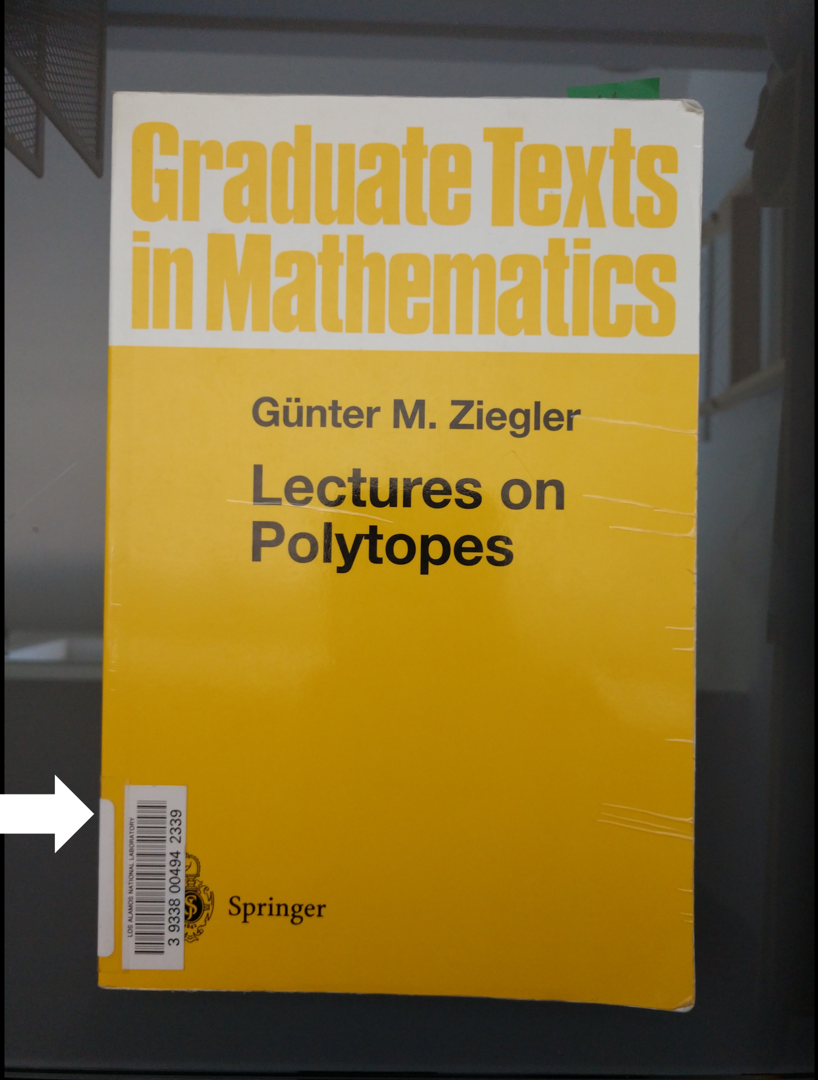
Fiorini, Massar, Pokutta, Tiwary, de Wolf -- Linear vs. Semidefinite Extended Formulations

This paper has *lots* to do with polytopes. It concludes with a
section called:

A Background on Polytopes

The paper mentions this book:

G. M. Ziegler.
Lectures on Polytopes,
volume 152 of Graduate Texts in Mathematics. Springer-Verlag, 1995.



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ColdQuanta

Rosetta stone: Ising models ⇔ Polytopes

Physics-speak

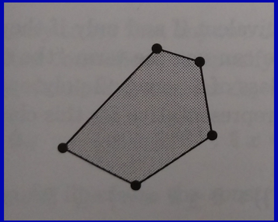
- Spin states
- Ising models / QUBOs
- Ground states

Math-speak

- $\mathcal{V}$-polytopes
- $\mathcal{H}$-polytopes
- Linear programming
- Face, Facet, Face lattice
- Cone, Normal Fan

Two ways to define polytopes

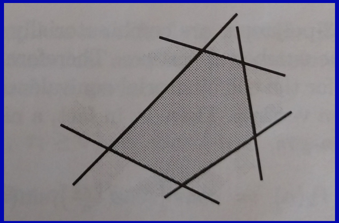
A *$\mathcal{V}$-polytope* is the convex hull of a finite set of points in some $\mathbb{R}^d$.



Ising model / QUBO

An *$\mathcal{H}$-polyhedron* is an intersection of finitely many closed halfspaces in some $\mathbb{R}^d$.

An *$\mathcal{H}$-polytope* is an *$\mathcal{H}$-polyhedron* that is bounded in the sense that it does not contain a ray $\{\boldsymbol{x} + t\boldsymbol{y} : t \geq 0\}$ for any $\boldsymbol{y} \neq 0$.



Linear programming

Mathematically equivalent
Algorithmically distinct

Faces, facets, face lattice

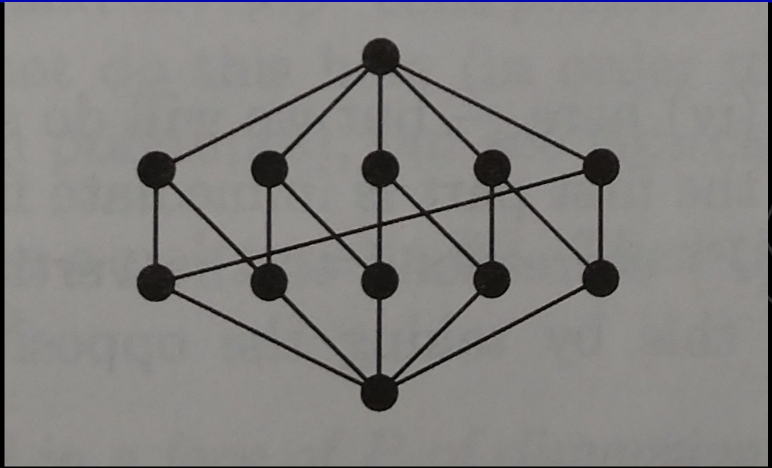
A *face* of P is any set of the form $F = P \cap \{x \in \mathbb{R}^d : cx = c_0\}$ where $cx \leq c_0$ is a valid inequality for P.

The *dimension* of a face is the dimension of its affine hull.

Faces of dimension dim(P)-1 are called *facets.*

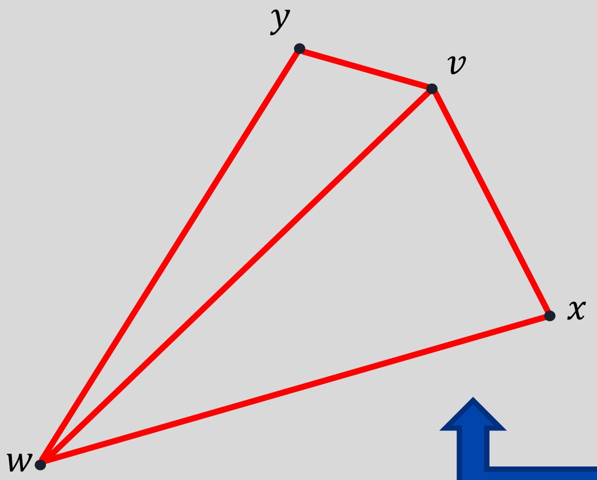
The *face lattice* of a convex polytope P is the poset of all faces of P partially ordered by inclusion.

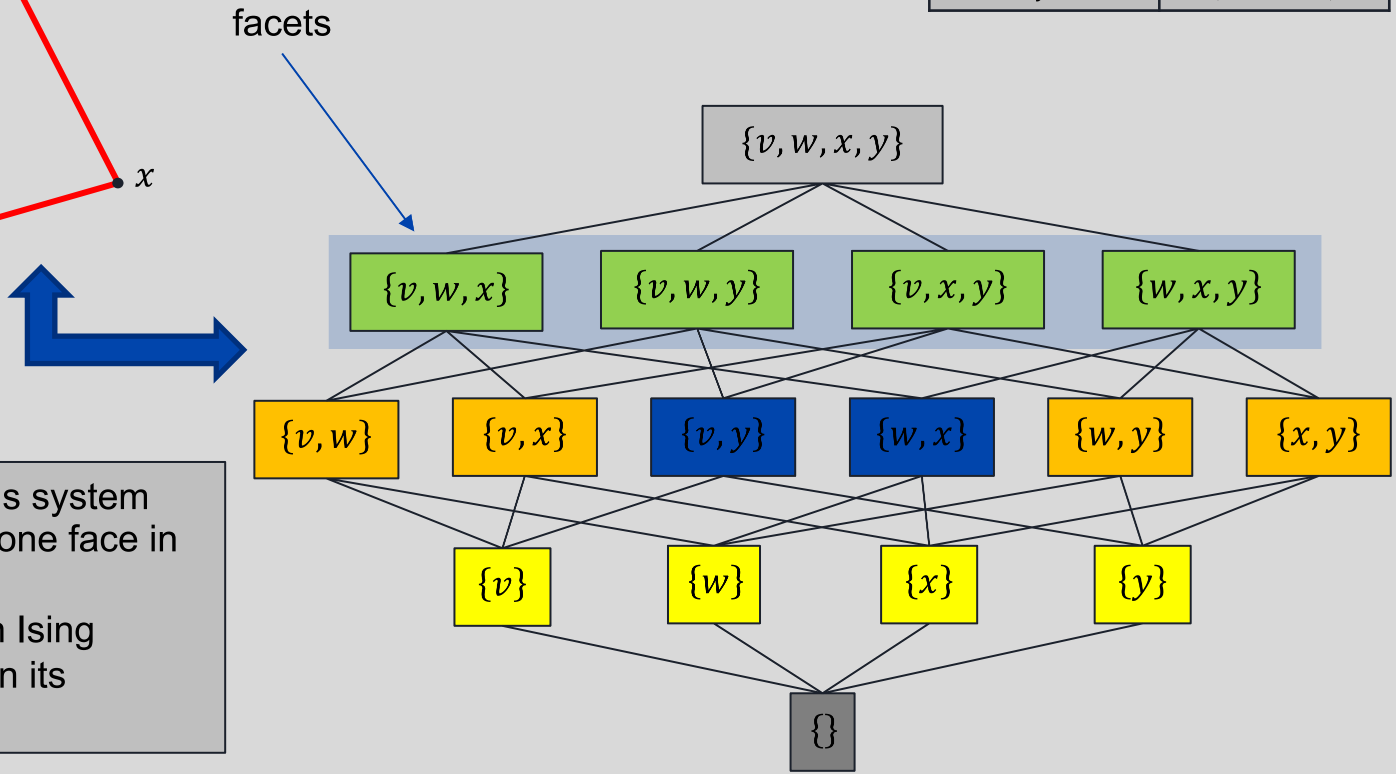
Face lattice of a convex pentagon



Face lattice of Chimeratope(1,1,1)
two spins, one coupler

vertex	$(\sigma_1, \sigma_2, \sigma_1\sigma_2)$
v	$(1,1,1)$
w	$(1,-1,-1)$
x	$(-1,1,-1)$
y	$(-1,-1,1)$



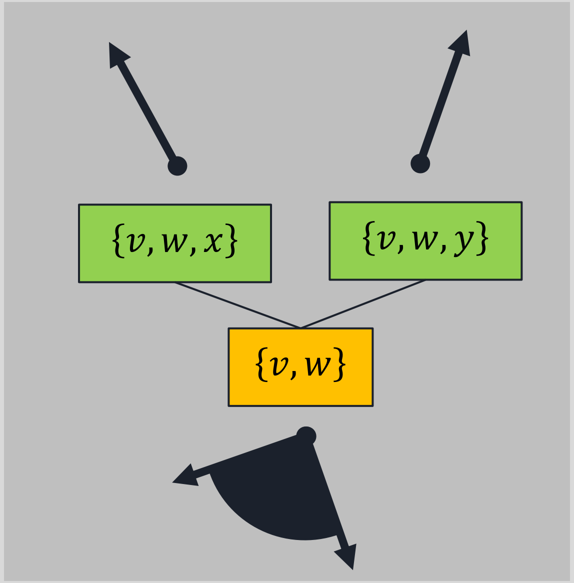


- Each Ising model for this system corresponds to exactly one face in this lattice.
- The ground states of an Ising model are the vertices in its corresponding face.

Example Ising models

Require $\alpha > 0$ and $\beta > 0$

Face	Ising model	Degeneracy
$\{v,w,x\}$	$\alpha(-1,+1,-1)\cdot(\sigma_1,\sigma_2,\sigma_1\sigma_2)$	3
$\{v,w,y\}$	$\beta(-1,+1,-1)\cdot(\sigma_1,\sigma_2,\sigma_1\sigma_2)$	3
$\{v,w\}$	$[\alpha(-1,+1,-1)+\beta(-1,+1,-1)]\cdot(\sigma_1,\sigma_2,\sigma_1\sigma_2)$	2



				$\{v,w,y\}$	$\{v,w,x\}$	$\{v,w\}$
vertex	σ_1	σ_2	$\sigma_1\sigma_2$	**energy**	**energy**	**energy**
v	1	1	1	$-\alpha$	$-\beta$	$-\alpha-\beta$
w	1	-1	-1	$-\alpha$	$-\beta$	$-\alpha-\beta$
x	-1	1	-1	3α	$-\beta$	$3\alpha-\beta$
y	-1	-1	1	$-\alpha$	3β	$-\alpha+3\beta$

Ising model symmetries: spin flip & automorphism

Examples at $L = 1$

Spin flip: $\sigma_1 \Rightarrow -\sigma_1$

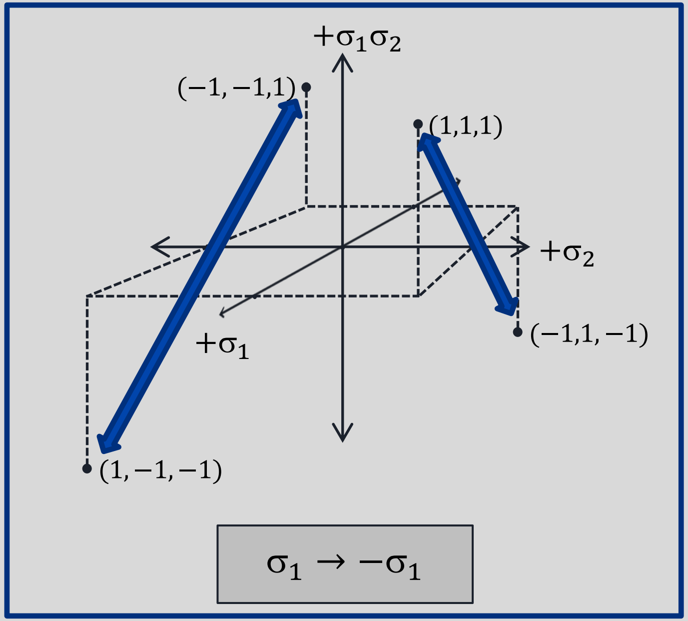
$$E = (h_1, h_2, J_{12}) \cdot (\sigma_1, \sigma_2, \sigma_1\sigma_2)$$
$$= (-h_1, h_2, -J_{12}) \cdot ((-\sigma_1), \sigma_2, (-\sigma_1)\sigma_2)$$

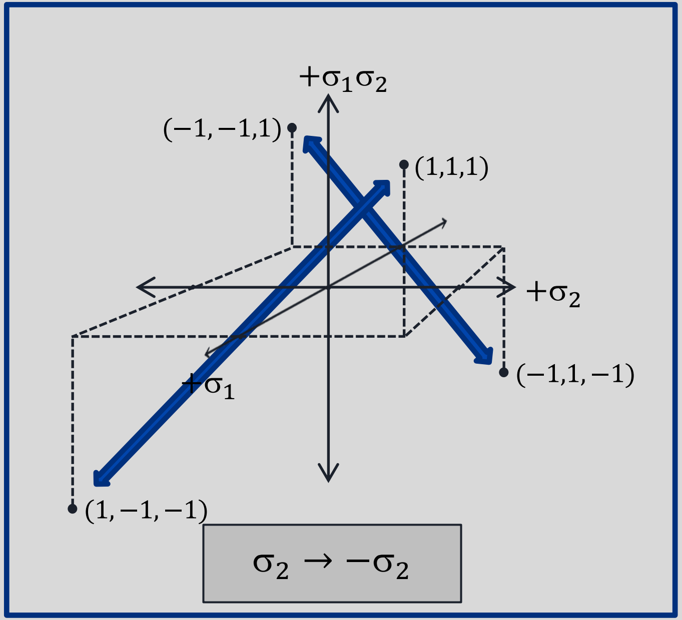
Automorphism: $(\sigma_1, \sigma_2) \Rightarrow (\sigma_2, \sigma_1)$

$$E = (h_1, h_2, J_{12}) \cdot (\sigma_1, \sigma_2, \sigma_1\sigma_2)$$
$$= (h_2, h_1, J_{12}) \cdot (\sigma_2, \sigma_1, \sigma_2\sigma_1)$$

- Fix M = N = 1
- Number of h_i coefficients: $2L$
- Number of J_{ij} coefficients: L^2
- Point cloud has 2^{2L} elements
- Point cloud is in $(2L + L^2)$-dimensional space
- Symmetry group:
  - Spin flip : $(Z_2)^{2L}$
  - Automorphism : $S_L \wr Z_2$ (wreath product)
  - Order of combined group : $2^{2L+1}(L!)^2$

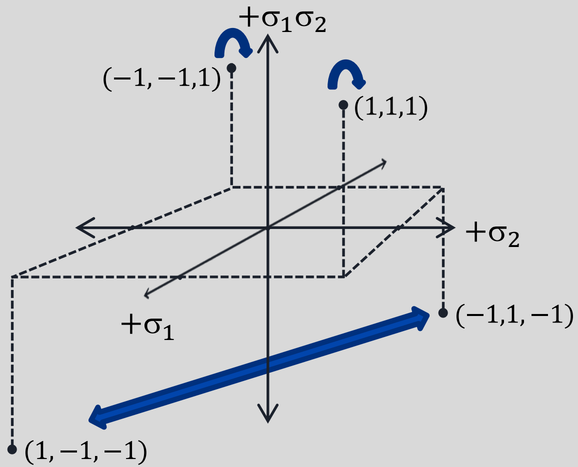
Spin flip symmetries of the L=1 Chimera unit cell





Automorphism symmetries of the L=1 Chimera unit cell

Transform spin vector and h/J vector so that energy (dot product) stays fixed



$(\sigma_1, \sigma_2) \rightarrow (\sigma_2, \sigma_1)$

Facet classes of the Chimeratope(L,1,1)

L	Dimension	Vertices	Facets	Facet classes
1	3	4	4	1
2	8	16	24	2
3	15	64	684	3
4	24	256	36391264	175

Equivalent Linear Program lives in a space of this dimensionality and is posed over a region defined by this many constraints

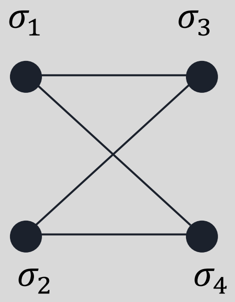
https://arxiv.org/pdf/1501.05407v4.pdf

Mathieu Dutour Sikirić

Faces of Chimeratope(2,1,1)

Dimension	Classes	Faces
8	1	1
7	2	24
6	5	168
5	9	520
4	11	816
3	12	712
2	6	360
1	4	104
0	1	16
-1	1	1

Convex polytope defined by 16 vertices in 8 dimensional space



Total number of inequivalent Ising models for this system is the sum over dimensions of classes: 2+5+...+1= 50

Cones, fans and normal fans

A *cone* is a nonempty set of vectors $C \in \mathbb{R}^d$ that contains any linear combination with nonnegative coefficients of a finite set of vectors from the cone.

Let P be a nonempty polytope in $\mathbb{R}^d$.

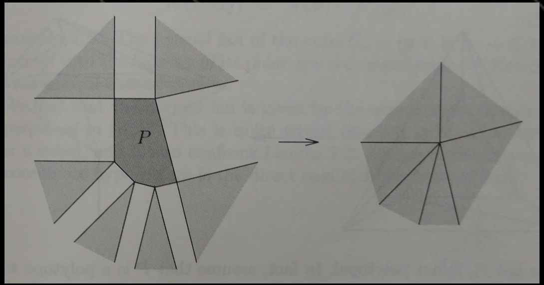
The *normal fan* of P is the set of cones of those linear functions which are maximal on a fixed face of P.

A *fan* in $\mathbb{R}^d$ is a family

$$\mathcal{F} = \{C_1, C_2, \ldots, C_N\}$$

of nonempty polyhedral cones, with the following two properties:

1) Every nonempty face of a cone in $\mathcal{F}$ is also a cone in $\mathcal{F}$
2) The intersection of any two cones in $\mathcal{F}$ is a face of both.



Relevance

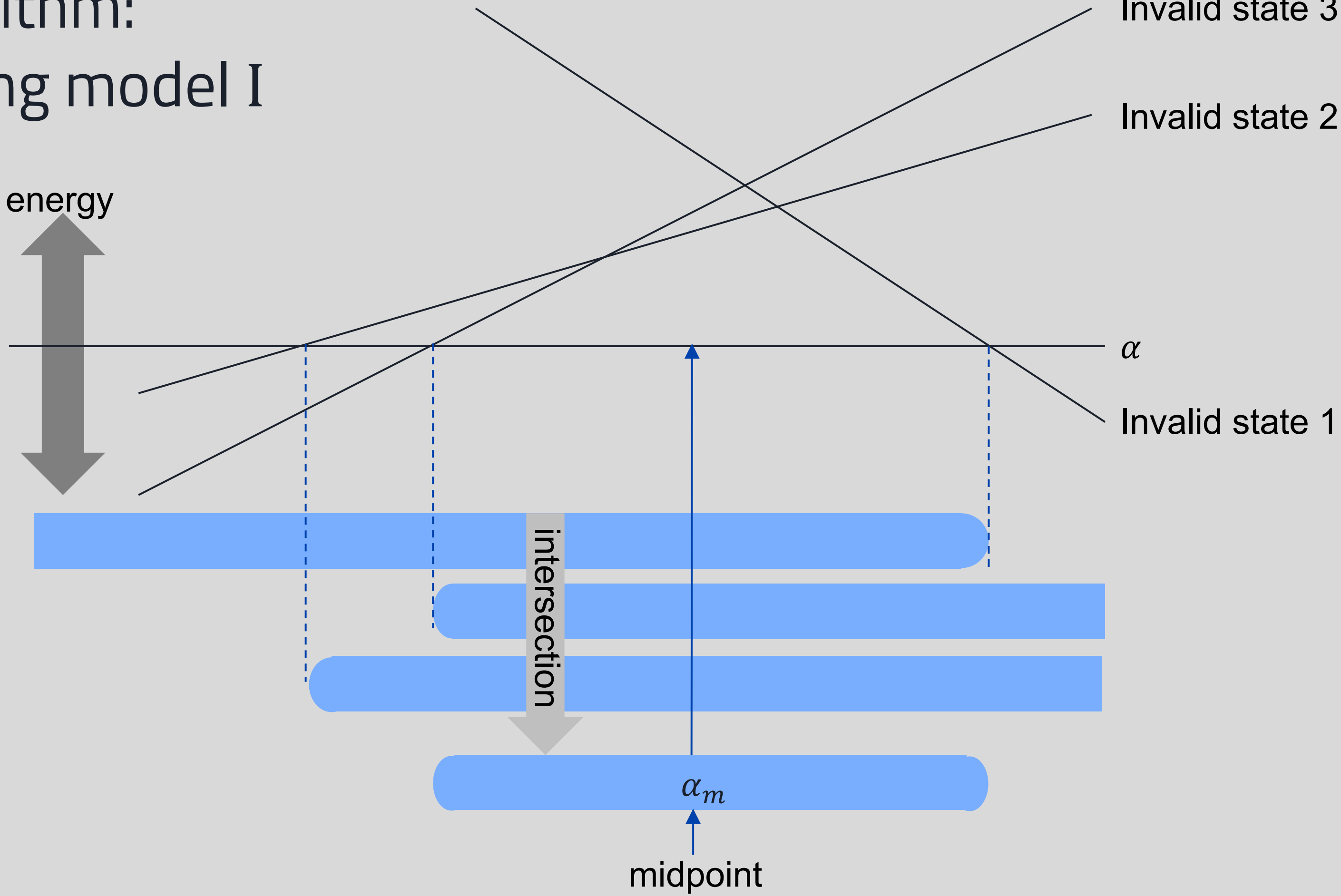
- Any Ising model for a fixed system picks out a specific set of ground states
- These ground states form a face of the polytope defined by the system
- This defines a map from Ising models to faces of the polytope
- The set of all Ising models mapping to a fixed face is a cone
- This set of cones can be organized into a normal fan

Why should we care?

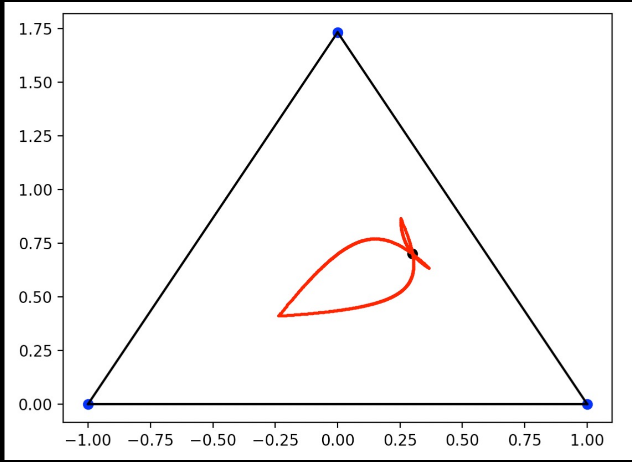
- All the Ising models in a cone are equivalent in the sense that they should produce the same ground states
- Because we are dealing with physical computational devices, all the Ising models in a cone are not equivalent.
- Some are better than others
- We should take advantage of this freedom to find better Ising models

Midpoint algorithm: improve an Ising model I

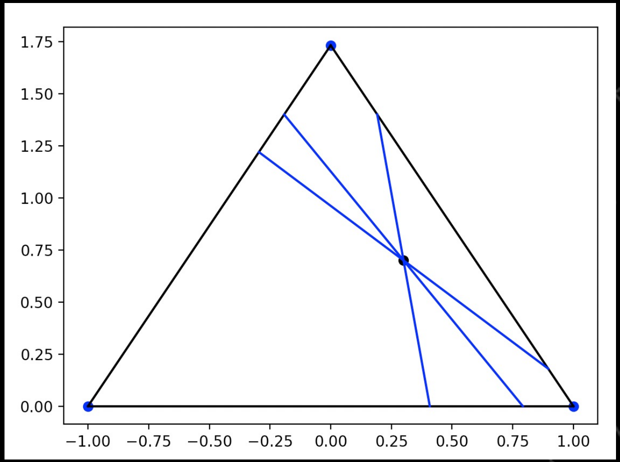
1. Pick a random Ising model Λ
2. Find δ such that $(\Lambda + \delta \mathrm{I}) \cdot \mathrm{I} = 0$
3. Define $\Lambda_\perp = \Lambda + \delta \mathrm{I}$
4. Compute the energy of each invalid state w.r.t. Ising model $I + \alpha\Lambda_\perp$ and determine α-interval where the state remains invalid
5. Form intersection of α-intervals and find midpoint α_m
6. $I \Leftarrow I + \alpha_m \Lambda_\perp$
7. Repeat



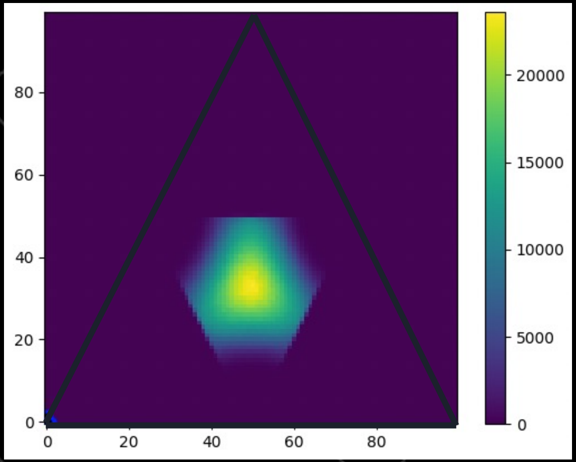
Midpoint algorithm example



An initial Ising model (black dot) can map to any of the Ising models indicated in red.



Any initial Ising model in blue can map to the final Ising model (black dot).



Density map of Ising models resulting from the midpoint algorithm.

Thank you!

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